

# On Perfect 2-Colorings of Johnson Graphs $J(v, 3)$

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**Abstract:** We study the perfect 2-colorings (also known as the equitable partitions into two parts or the completely regular codes with covering radius 1) of the Johnson graphs  $J(v, 3)$ . In particular, we classify all the realizable quotient matrices of perfect 2-colorings for odd  $v$ .

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## 1. INTRODUCTION

A *perfect coloring*<sup>1</sup> (or an *equitable partition*) of a graph  $\Gamma$  with  $t$  colors (perfect  $t$ -coloring for short) is a partition of the vertex set of  $\Gamma$  into  $t$  parts (colors)  $P_1, \dots, P_t$  such that, for all  $i, j \in \{1, \dots, t\}$ , every vertex of  $P_i$  is adjacent to the same number of vertices, namely,  $p_{ij}$  vertices, of  $P_j$ . The matrix  $P := (p_{ij})_{i,j=1,\dots,t}$  is called the *quotient matrix* of the perfect  $t$ -coloring. We do not distinguish between colorings obtained by renaming the colors (i.e., by equal permutations of rows and columns of  $P$ ). Note that every eigenvalue of  $P$  is that of  $\Gamma$ , see [8].

The *Johnson graph*  $J(v, k)$  (without loss of generality, let  $2k \leq v$ ) is a graph whose vertex set consists of all  $k$ -subsets of a fixed  $v$ -set; two  $k$ -sets are adjacent if and only if

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<sup>1</sup>This term is due to D. Fon-Der-Flaass, cf. [7].

they share  $k - 1$  elements exactly. The distinct eigenvalues of  $J(v, k)$  are the numbers  $\theta_i := (k - i)(v - k - i) - i$ ,  $i = 0, \dots, k$ , see [4, Chapter 9.1].

Given a perfect 2-coloring of the Johnson graph  $J(v, k)$ , a set  $C$  of all vertices of the same color is a completely regular code with covering radius 1. (For the definitions and background, see Section 2.) The quotient matrix  $P$  of a perfect 2-coloring has two distinct eigenvalues that are those of  $J(v, k)$ :  $\theta_0$  and  $\theta_s$ ,  $s \in \{1, 2, \dots, k\}$  (so that the strength of the corresponding completely regular code equals  $s - 1$ ). The completely regular codes with strength zero in  $J(v, k)$  were classified by Meyerowitz in [14]. If  $s = k$  then the vertices of the same color are the blocks of a  $(k - 1)$ -( $v, k, \lambda$ )-design and, conversely, the blocks of an arbitrary  $(k - 1)$ -( $v, k, \lambda$ )-design as the vertices of  $J(v, k)$  give a completely regular code with covering radius 1 and strength  $k - 1$ , see [11]. (Recall that, for a 2-design with  $k = 3$ , the necessary conditions are known to be sufficient [5].)

In this paper, we study the perfect 2-colorings of the Johnson graphs  $J(v, 3)$ . By the above, we are mainly interested in the case  $s = 2$ . For odd  $v$ , we show that there are no such 2-colorings, see Theorem 3.1. In case of even  $v$ , we consider the perfect 2-colorings with symmetric quotient matrix. Then one can show that  $v$  must be congruent to 2 modulo 4. There are examples of such 2-colorings when  $v \in \{6, 10\}$ . However, for  $v > 10$ , we prove that there are no such 2-colorings, see Theorem 3.2.

The paper is organized as follows. In Sections 2.1 and 2.2, we recall the various notions and results concerning graphs and their codes. In Section 2.3, we give the background of the present work. In Section 3, we describe our approach of investigation of perfect 2-colorings in  $J(v, 3)$  and prove our main results. Section 4 contains some concluding remarks concerning future results.

## 2. DEFINITIONS AND PRELIMINARIES

### 2.1. Graphs

We consider only finite undirected graphs without loops or multiple edges. Let  $\Gamma$  be a connected graph. The *distance*  $d(x, y)$  between any two vertices  $x$  and  $y$  of  $\Gamma$  is the length of a shortest path from  $x$  to  $y$  in  $\Gamma$ . The *diameter*  $\text{diam}(\Gamma)$  of  $\Gamma$  is the maximal distance occurring in  $\Gamma$ .

For a subset  $X$  of the vertex set of  $\Gamma$ , we will also write  $X$  for the subgraph of  $\Gamma$  induced by  $X$ . For a vertex  $x$  of  $\Gamma$ , define  $\Gamma_i(x)$  to be the set of vertices that are at distance  $i$  from  $x$  ( $i = 0, \dots, \text{diam}(\Gamma)$ ). The subgraph  $\Gamma_1(x)$  is called the *neighborhood* of a vertex  $x$ . We will write  $\Gamma(x)$  instead of  $\Gamma_1(x)$  for short.

A connected graph  $\Gamma$  with diameter  $d := \text{diam}(\Gamma)$  is *distance-regular* if there are integers  $b_i, c_i$  ( $i = 0, \dots, d$ ) such that, for every pair of vertices  $x, y \in \Gamma$  with  $d(x, y) = i$ , there are exactly  $c_i$  neighbors of  $x$  in  $\Gamma_{i-1}(y)$  and  $b_i$  neighbors of  $x$  in  $\Gamma_{i+1}(y)$  (we assume that  $\Gamma_{-1}(y)$  and  $\Gamma_{d+1}(y)$  are empty sets). We also define  $a_i := b_0 - b_i - c_i$ . The numbers  $a_i, b_i, c_i$  ( $i = 0, \dots, d$ ) are called the *intersection numbers* and the array  $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$  is called the *intersection array* of the distance-regular graph  $\Gamma$ .

By an *eigenvalue* of a graph  $\Gamma$ , we mean an eigenvalue of its adjacency matrix. We recall that a distance-regular graph with diameter  $d$  has  $d + 1$  distinct eigenvalues exactly, which can be calculated from its intersection array, see [4, Section 4.1.B].

Let  $\Gamma$  be a distance-regular graph with diameter  $d \geq 2$ . For  $i = 0, \dots, d$ , define  $A_i$  to be a square  $(0, 1)$ -matrix of size  $|\Gamma|$  whose rows and columns are indexed by the vertex set of  $\Gamma$ , and, for all  $x, y \in \Gamma$ , set  $(A_i)_{x,y} := 1$  if and only if  $d(x, y) = i$ . In particular,  $A_1$  is just the adjacency matrix of  $\Gamma$ . Since  $\Gamma$  is distance-regular, we see that, for  $i = 0, \dots, d$ ,

$$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1},$$

and this implies [4] that  $A_i = p_i(A_1)$  for certain polynomial  $p_i$  of degree  $i$ . Hence,  $A_1$  generates the matrix algebra  $\mathcal{A}$  over  $\mathbb{C}$ , the so-called *Bose–Mesner algebra*, of dimension  $d + 1$ , and the set of matrices  $A_0 = I, A_1, \dots, A_d$  is a basis of  $\mathcal{A}$ . Since the algebra  $\mathcal{A}$  is semisimple and commutative,  $\mathcal{A}$  also has a basis of pairwise orthogonal idempotents  $E_0 := \frac{1}{|\Gamma|} J, E_1, \dots, E_d$  [4] (the so-called *primitive idempotents* of  $\mathcal{A}$ ). In fact,  $E_j$  is the matrix representing orthogonal projection onto the eigenspace of  $A_1$  corresponding to some eigenvalue of  $\Gamma$ . For the distinct eigenvalues  $b_0 = \theta_0 > \theta_1 > \dots > \theta_d$  of  $\Gamma$ , the basis of primitive idempotents is usually ordered so that

$$A_1 E_j = \theta_j E_j, j = 0, \dots, d.$$

An  $l$ -clique  $L$  of  $\Gamma$  is a complete subgraph (i.e., every two vertices of  $L$  are adjacent) of  $\Gamma$  with exactly  $l$  vertices. We say that  $L$  is a clique if it is an  $l$ -clique for certain  $l$ .

By the  $n \times m$ -grid, we mean the Cartesian product of two complete graphs on  $n$  and  $m$  vertices.

The *Johnson graph*  $J(v, k)$  is distance-regular with diameter  $k$ , the following intersection numbers:

$$b_{i-1} := (k - (i - 1))(v - k - (i - 1)), c_i := i^2, i = 1, \dots, k,$$

and eigenvalues

$$\theta_i := (k - i)(v - k - i) - i, i = 0, \dots, k.$$

The proof of Lemma 2.1 is straightforward.

**Lemma 2.1.** *The following holds.*

- (1) *for a vertex  $x$  of  $J(v, k)$ , the neighborhood of  $x$  is the  $k \times (v - k)$ -grid;*
- (2) *for every pair of vertices  $x, y$  at distance 2 in  $J(v, k)$ , the subgraph induced by their common neighbors is a 4-cycle;*
- (3) *for a vertex  $x$  of  $J(v, k)$  and for every 4-cycle  $C$  in its neighborhood, there is a unique vertex  $y$  such that  $d(x, y) = 2$  and  $C$  is the subgraph induced by the common neighbors of  $x$  and  $y$ .*

## 2.2. Codes in Graphs

Let  $\Gamma$  be a graph. An arbitrary subset  $C$  of the vertex set of  $\Gamma$  is called a *code* in graph  $\Gamma$ .

For a code  $C \subseteq \Gamma$  with  $|C| \geq 2$ , we define the *minimum distance* of  $C$ :

$$\delta_C := \min\{d(x, y) | x, y \in C, x \neq y\},$$

and the *covering radius*:

$$\rho_C := \max\{d(x, C) | x \in \Gamma\},$$

where, for a vertex  $x \in \Gamma$ ,  $d(x, C) := \min\{d(x, y) | y \in C\}$ .

A code  $C$  is called an *e-code* if  $e$  is a maximal integer such that  $\delta_C \geq 2e + 1$ , i.e., balls with radius  $e$  around the vertices of  $C$  are all pairwise disjoint. If  $\Gamma$  is distance-regular then, for any its  $e$ -code  $C$  and a vertex  $x \in C$ , the so-called *sphere packing bound* holds [8]:

$$|C| \sum_{i=0}^e |\Gamma_i(x)| \leq |\Gamma|.$$

An  $e$ -code  $C$  is called *perfect* if equality holds in this bound (*non-trivial* if  $e > 0$  and  $|C| > 2$ ). Evidently, in the Hamming graphs (see [4, Chapter 9]) this notion is equivalent to that in the coding theory [19].

The error-correcting perfect codes over finite fields were classified by Zinoviev and Leontiev [10], and Tietäväinen [18]. In [6], Delsarte introduced completely regular codes — a class of codes with nice combinatorial properties similar to those observed in perfect codes. In particular, the class of completely regular codes includes all perfect codes. Delsarte gave the definition not only for codes in Hamming graphs, but also for codes in arbitrary distance-regular graphs, and conjectured the nonexistence of nontrivial perfect codes in Johnson graphs. All presently known results [9] confirm this conjecture.

A code  $C$  gives a natural partition of the vertex set of  $\Gamma$  according to distance from  $C$ : for  $i = 0, \dots, \rho_C$ , define  $\Gamma_i(C)$  to be the set of vertices that are at distance  $i$  from  $C$ . The partition  $\{C = \Gamma_0(C), \Gamma_1(C), \dots, \Gamma_{\rho_C}(C)\}$  will be referred to as the *distance partition* of the vertex set of  $\Gamma$  with respect to  $C$ . For a vertex  $x$  and a code  $C$ , we write  $\delta_i(x, C) := |\Gamma_i(x) \cap C|$ . The numbers  $\delta_i(x, C)$ ,  $i = 0, \dots, \text{diam}(\Gamma)$ , are called the *outer distribution numbers* of  $C$ .

Following Delsarte [6], we say that a code  $C$  in a distance-regular graph  $\Gamma$  is *completely regular* if the outer distribution number  $\delta_i(x, C)$  only depends on  $i$  and  $d(x, C)$ , i.e., there exist integers  $\delta_{il}$  ( $i = 0, \dots, \text{diam}(\Gamma)$ ,  $l = 0, \dots, \rho_C$ ) such that, for every vertex  $x \in \Gamma$  and all  $i = 0, \dots, \text{diam}(\Gamma)$ ,  $\delta_i(x, C) = \delta_{il}$  holds, where  $l := d(x, C)$ .

Let  $\Gamma$  be a distance-regular graph and  $C$  be a code with covering radius  $\rho$  in  $\Gamma$ . Consider the distance partition  $\Pi := \{C = \Gamma_0(C), \Gamma_1(C), \dots, \Gamma_{\rho}(C)\}$  of the vertex set of  $\Gamma$  w.r.t.  $C$ . It follows from [17] that  $C$  is completely regular if and only if  $\Pi$  is a perfect  $(\rho + 1)$ -coloring. In this case the quotient matrix  $P$  of the partition  $\Pi$  can be written in the following tridiagonal form:

$$P = \begin{pmatrix} \alpha_0 & \beta_0 & 0 & \dots & 0 \\ \gamma_1 & \alpha_1 & \beta_1 & & \\ 0 & \gamma_2 & \alpha_2 & \beta_2 & \\ \vdots & & \ddots & \ddots & \ddots \\ 0 & \dots & 0 & \gamma_{\rho} & \alpha_{\rho} \end{pmatrix},$$

where  $\alpha_i, \beta_i, \gamma_i$  ( $i = 0, \dots, \rho$ ) are integers such that each vertex in  $\Gamma_i(C)$  has  $\alpha_i$  neighbors in  $\Gamma_i(C)$ ,  $\beta_i$  neighbors in  $\Gamma_{i+1}(C)$ , and  $\gamma_i$  neighbors in  $\Gamma_{i-1}(C)$ . These numbers are called the *intersection numbers* of the completely regular code  $C$ .

Let us remark that the quotient matrix  $P$  has  $\rho + 1$  distinct eigenvalues exactly; each of them is an eigenvalue of  $\Gamma$ , see [8]. In addition, the set  $\Gamma_\rho(C)$  is also completely regular, see [17] (so that in the case  $\rho = 1$  both  $C = \Gamma_0(C)$  and  $\Gamma_1(C) = \Gamma \setminus C$  are completely regular).

Further note that  $b_0 = \theta_0$  is always an eigenvalue of  $P$ . Let  $\chi_C$  be the characteristic vector of  $C$  (here  $\chi_C$  is indexed by the vertex set of  $\Gamma$ ). The following lemma will be used in Section 2.3.

**Lemma 2.2** ([13, Lemma 1]). *Let  $E_0, E_1, \dots, E_d$  be the primitive idempotents of  $\Gamma$ . Then, for  $j = 1, \dots, d$ ,  $E_j \chi_C \neq \mathbf{0}$  holds if and only if  $\theta_j$  is an eigenvalue of  $P$ .*

For more results and background on completely regular codes (in various graphs) we refer the reader to [4] and the survey [13]. From now on, we restrict our attention to the Johnson graphs  $J(v, k)$  and their completely regular codes.

### 2.3. Completely Regular Codes in Johnson Graphs

Recall that a  $t$ -( $v, k, \lambda$ )-*design* ( $t$ -*design* for short) is a collection of  $k$ -subsets (called *blocks*) of a fixed  $v$ -set such that every  $t$ -subset occurs in  $\lambda$  blocks exactly (in this paper, repeated blocks are not allowed in a design). Note that a 0-design is just a collection of  $k$ -subsets. The *strength* of a design  $D$  is the largest  $t$  such that  $D$  is a  $t$ -design.

By the definition of the Johnson graph  $J(v, k)$ , we may consider an arbitrary subset  $C$  of its vertices as a collection of some  $k$ -subsets of a fixed  $v$ -set  $X$ . Let  $\chi_C$  be the characteristic vector of a subset  $C$  of the vertex set of  $J(v, k)$  and  $E_0, E_1, \dots, E_k$  be the primitive idempotents of  $J(v, k)$  (recall that  $k \leq v/2$  and  $k = \text{diam}(J(v, k))$ ). The following important result is due to Delsarte.

**Theorem 2.3** ([6, Theorem 4.2]). *A subset  $C$  of the vertex set of  $J(v, k)$  is a  $t$ -design if and only if*

$$E_1 \chi_C = E_2 \chi_C = \dots = E_t \chi_C = \mathbf{0}.$$

It now follows from Lemma 2.2 and Theorem 2.3 that a subset  $C$  of vertices in  $J(v, k)$ , being a completely regular code, is a  $t$ -design. The strength of this design is the smallest  $t$  such that  $\theta_{t+1}$  is an eigenvalue of  $P$ . (For this reason, a completely regular code in  $J(v, k)$  is sometimes referred to as a *completely regular design*, see [12].) In the case  $\rho = 1$ , we have the following convenient corollary.

**Lemma 2.4** ([2, Theorem 1, Corollary 1]). *Let  $C$  be a completely regular code with covering radius 1 and quotient matrix  $P := (p_{ij})_{2 \times 2}$  in  $J(v, k)$ . Then  $p_{11} + p_{12} = \theta_0$ ,  $p_{11} - p_{21} = \theta_{t+1}$  for some  $t \in \{0, \dots, k-1\}$ , and the vertices of  $C$  are the blocks of a design with parameters*

$$t - \left( v, k, \frac{p_{21}}{p_{12} + p_{21}} \binom{v-t}{k-t} \right).$$

(Therefore, for all  $0 \leq j \leq i \leq t$ , the numbers

$$\frac{p_{21}}{p_{12} + p_{21}} \binom{v-i}{k-i+j}$$

are integer.)

Let us recall some results on completely regular codes with small strength in  $J(v, k)$ . The completely regular codes with strength 0 in Johnson graphs were classified by Meyerowitz.

**Theorem 2.5** ([14, Theorem J]). *Let  $C$  be a completely regular design in  $J(v, k)$  with zero strength. Then there is a subset  $Y \subset X$  such that*

$$\text{either } C = \{c \subset X : |c| = k, Y \subseteq c\} \quad \text{or} \quad C = \{c \subset X : |c| = k, c \subseteq Y\}.$$

Following the approach that was used by Meyerowitz to prove Theorem 2.5, Martin [11] found all completely regular codes in  $J(v, k)$  with strength 1 and minimum distance at least 2. Let us describe his result. Suppose that  $|X| = v = qp$ ,  $k = sp$ , where  $q \geq 2s$ ,  $p \geq 2$ , and consider a partition  $X = X_1 \dot{\cup} X_2 \dot{\cup} \dots \dot{\cup} X_q$  into  $q$  groups, each of size  $p$ . Define  $C$  to be the set of all vertices of the form  $\cup_{i \in I} X_i$ , where  $I$  runs over all  $s$ -element subsets of  $\{1, 2, \dots, q\}$  (so that  $|C| = \binom{q}{s}$ ). Then  $C$  is a design with strength 1 and minimum distance  $p \geq 2$ . Martin [11] called such a design a *groupwise complete design* and showed that a groupwise complete design is completely regular if and only if one of the following holds:

- $p = k, v = 2k$ ,
- $p = 2$ ,
- $p = 3, s = 1$ .

Note that the covering radius of a groupwise complete design is at least 2.

**Theorem 2.6** ([11, Theorem 3.1]). *Let  $C$  be a completely regular design in  $J(v, k)$  having strength 1 and minimum distance at least 2. Then  $C$  is a groupwise complete design.*

We should also mention one more result by Martin.

**Theorem 2.7** ([12, Corollary 3.5]). *Let  $C$  be a  $(k-1)$ -design in  $J(v, k)$ . Then  $C$  is completely regular.*

We now turn to completely regular codes with smallest covering radius in  $J(v, k)$ , i.e., perfect 2-colorings. (Note that the problem of existence of those includes the Delsarte conjecture, see [3], [13].) Perfect 2-colorings were studied by Avgustinovich and Mogilnykh in several papers [1], [2], [3], [15], [16]. Let us survey some of their results. For a distance-regular graph  $\Gamma$ , let  $\{P_1, P_2\}$  be a perfect 2-coloring of  $\Gamma$  with quotient matrix  $P$  and  $C$  be a completely regular code in  $\Gamma$  with covering radius  $\rho$  and intersection numbers  $\alpha_i, \beta_i, \gamma_i$  ( $i = 0, \dots, \rho$ ). The following theorem can be considered as a generalization of that from [17].

**Theorem 2.8** ([1, Theorem 1]). *For all  $0 \leq i \leq \rho$ , the following holds:*

$$|P_1 \cap \Gamma_{i-1}(C)|\beta_{i-1} + |P_1 \cap \Gamma_i(C)|(p_{21} - p_{11} + \alpha_i) + |P_1 \cap \Gamma_{i+1}(C)|\gamma_{i+1} = |\Gamma_i(C)|p_{21}.$$

Suppose that a distance-regular graph  $\Gamma$  with diameter  $d$  is *antipodal*, i.e., the relation *being at distance  $d$  or 0* induces an equivalence relation on the vertex set of  $\Gamma$ , and, in addition, suppose that an equivalence class contains two vertices exactly (and note that  $J(2m, m)$  satisfies this condition). Given a perfect 2-coloring of  $\Gamma$ , the two vertices from one equivalence class may be colored with one color or two different colors. But Theorem 2.8 implies that only one case may appear in a given perfect 2-coloring of  $\Gamma$ . Moreover, in the second case (when the vertices from every equivalence class are colored with two different colors) the quotient matrix is symmetric.

In [3], two infinite series of perfect 2-colorings of  $J(v, 4)$  and  $J(v, 5)$  based on Steiner triple and quadruple systems were found. In [2], three perfect 2-colorings of  $J(2m, 3)$  with quotient matrices

$$\begin{pmatrix} 3(2m-5) & 6 \\ 4(m-2) & 2m-1 \end{pmatrix}, \begin{pmatrix} 3(m-3) & 3m \\ m-2 & 5m-7 \end{pmatrix}, \text{ and } \begin{pmatrix} 3(m-1) & 3(m-2) \\ m+4 & 5m-13 \end{pmatrix} \quad (1)$$

were constructed with merging some orbits of an automorphisms group of  $J(2m, 3)$ . Note that these matrices have the second eigenvalue  $2m - 7 = \theta_2$  of  $J(2m, 3)$  (so that the corresponding designs have strength 1).

Also, in [1], [2], [16], all realizable quotient matrices of perfect 2-colorings of  $J(v, k)$ , where  $v \leq 8$ , were listed and, for  $J(9, 3)$ , the existence of a perfect 2-coloring with quotient matrix  $\begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix}$  was left as an open case.

**Theorem 2.9** ([1, Theorem 5]). *The following matrices are all the realizable quotient matrices of perfect 2-colorings of  $J(7, 3)$ :*

$$\begin{pmatrix} 9 & 3 \\ 4 & 8 \end{pmatrix}, \begin{pmatrix} 0 & 12 \\ 3 & 9 \end{pmatrix}, \begin{pmatrix} 3 & 9 \\ 6 & 6 \end{pmatrix}.$$

In the next section, we shall give a general answer for  $J(v, 3)$  with odd  $v$ .

### 3. PERFECT 2-COLORINGS OF JOHNSON GRAPHS $J(v, 3)$

In this section, we denote a graph  $J(v, 3)$  by  $\Gamma$ . Let  $\{P_1, P_2\}$  be a perfect 2-coloring of  $\Gamma$  with quotient matrix  $P$ ,

$$P := \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix},$$

(in other words,  $P_1$  and  $P_2$  are completely regular codes with covering radius 1 in  $\Gamma$ ). Then  $P$  has exactly two distinct eigenvalues that are those of  $J(v, 3)$ :  $\theta_0 = 3(v-3)$  and  $\theta_s = (3-s)(v-3-s) - s$  for some  $s \in \{1, 2, 3\}$ . Now Theorem 2.5 allows us to

assume that  $s > 1$ . If  $s = 3$  then, by Lemma 2.4 and Theorem 2.7, the vertices of  $P_i$ ,  $\{i, j\} = \{1, 2\}$ , are the blocks of a 2-design with parameters

$$2 - \left( v, 3, \frac{p_{ji}}{p_{ij} + p_{ji}}(v - 2) \right),$$

and, conversely, the blocks of every such a design give a completely regular code with covering radius 1 in  $J(v, 3)$ . Further, it follows from [5] that the following congruences:

$$\lambda(v - 1) \equiv 0 \pmod{2}, \lambda v(v - 1) \equiv 0 \pmod{6}$$

are necessary and sufficient conditions for a  $2-(v, 3, \lambda)$ -design to exist. Therefore, we are mainly interested in the case  $s = 2$ .

In this section, we prove the following theorems.

**Theorem 3.1.** *Let  $v$  be odd and  $\{P_1, P_2\}$  be a perfect 2-coloring of  $J(v, 3)$ . Then the vertices of  $P_i$ ,  $i \in \{1, 2\}$ , are the blocks of a design with strength 0 or 2. In other words, there are no perfect 2-colorings of  $J(v, 3)$  with quotient matrix  $P$  such that  $\theta_2$  is an eigenvalue of  $P$ .*

**Theorem 3.2.** *Let  $\{P_1, P_2\}$  be a perfect 2-coloring of  $J(v, 3)$  with symmetric quotient matrix  $P$  such that  $\theta_2$  is an eigenvalue of  $P$ . Then  $P = \begin{pmatrix} 2v - 8 & v - 1 \\ v - 1 & 2v - 8 \end{pmatrix}$ , and  $v \in \{6, 10\}$ .*

Let  $y$  be an arbitrary vertex of  $\Gamma$ . The perfect 2-coloring  $\{P_1, P_2\}$  naturally induces a partition of  $\{y\} \cup \Gamma(y)$  into two parts. The main idea of the proof is to consider such a partition and establish some of its properties and connections with similar partitions for vertices of  $\Gamma(y)$ . Roughly speaking, for a given partition of  $\{y\} \cup \Gamma(y)$  and a vertex  $x \in \Gamma(y)$ , we will be able to derive a partition of  $\{x\} \cup \Gamma(x)$ .

In what follows, we suppose that  $P$  has an eigenvalue  $\theta \neq \theta_0$  of  $J(v, 3)$  (not necessarily  $\theta = \theta_2$ ). For a subset  $S$  of the vertex set of  $\Gamma$ , define

$$\overline{S} := |S \cap P_1|.$$

To simplify the notation, we use  $\overline{x}$  instead of  $\overline{\{x\}}$  if  $x$  is a vertex of  $\Gamma$ , i.e.,

$$\overline{x} := \begin{cases} 1 & \text{if } x \in P_1, \\ 0 & \text{if } x \in P_2. \end{cases}$$

Note that

$$\overline{\Gamma(x)} = p_{11}\overline{x} + p_{21}(1 - \overline{x}) = \theta\overline{x} + p_{21}. \quad (2)$$

Recall that by Lemma 2.1, for a vertex  $y \in \Gamma$ , the subgraph  $\Gamma(y)$  is the  $3 \times (v - 3)$ -grid. So that every maximal clique in  $\Gamma$  has size 4 or  $v - 2$ . If  $X$  is an underlying  $v$ -element set for  $\Gamma$ , i.e., the vertices of  $\Gamma$  are all the 3-element subsets of  $X$ , then a



maximal 4-clique is induced by the 3-element subsets whose union contains exactly four elements of  $X$ ; a maximal  $(v - 2)$ -clique is induced by the 3-element subsets that contain two fixed elements of  $X$ . In the  $3 \times (v - 3)$ -grid, we call a maximal  $(v - 3)$ -clique a *row*, and a maximal 3-clique a *column*.

For a vertex  $x \in \Gamma(y)$ , we denote by  $(y, x]$  the row of  $\Gamma(y)$  that contains  $x$ , and by  $[y, x]$  the maximal  $(v - 2)$ -clique that contains  $x, y$ . Note that  $y \notin (y, x]$  and  $[y, x] = \{y\} \cup (y, x]$ .

Let  $\{x_1, x_2, x_3\}$  induce a column of  $\Gamma(y)$ . Then, for  $\{i, j, k\} = \{1, 2, 3\}$ , a vertex  $x_i$  lies in a column (namely,  $\{y, x_i, x_k\}$ ) and a row of  $\Gamma(x_j)$ . Thus, the vertices  $x_i$  and  $x_j$  have exactly  $v - 4$  common neighbors in  $\Gamma_2(y)$ , which induce a clique. We denote by  $(x_i, x_j)$  these common neighbors, i.e.,

$$(x_i, x_j) := [x_i, x_j] \setminus \{x_i, x_j\} = (x_i, x_j] \setminus \{x_j\} = (x_j, x_i] \setminus \{x_i\} = \Gamma(x_i) \cap \Gamma(x_j) \setminus \Gamma(y),$$

$$(x_i, x_j) = (x_j, x_i).$$

For a vertex  $y \in \Gamma$ , let  $\{x_{i\delta} | i = 1, 2, 3, \delta = 1, \dots, v - 3\}$  be the vertex set of  $\Gamma(y)$  (so that distinct vertices  $x_{i\delta}$  and  $x_{j\epsilon}$  are adjacent if and only if  $i = j$  or  $\delta = \epsilon$ ). Then define a matrix

$$M(y) := \begin{pmatrix} \bar{x}_{1,1} & \bar{x}_{1,2} & \dots & \bar{x}_{1,v-3} \\ \bar{x}_{2,1} & \bar{x}_{2,2} & \dots & \bar{x}_{2,v-3} \\ \bar{x}_{3,1} & \bar{x}_{3,2} & \dots & \bar{x}_{3,v-3} \end{pmatrix}.$$

For two matrices  $M_1$  and  $M_2$  of equal size, we write  $M_1 \sim M_2$  if they can be obtained from each other by some permutations of rows and columns. Otherwise, we write  $M_1 \not\sim M_2$ . For a matrix  $M$  and a set of matrices  $\mathcal{M} := \{M_1, \dots, M_m\}$ , we write  $M \rightsquigarrow \mathcal{M}$  if  $M \sim M_i$  for at least one matrix  $M_i \in \mathcal{M}$  (otherwise, we write  $M \not\rightsquigarrow \mathcal{M}$ ).

**Lemma 3.3.** *For a vertex  $y \in \Gamma$  and a column  $\{x_1, x_2, x_3\}$  of  $\Gamma(y)$ , the following holds:*

$$\overline{(x_i, x_j)} - \overline{(y, x_k]} = \frac{\theta + 1}{2}(\bar{x}_i + \bar{x}_j - \bar{x}_k - \bar{y}) - \bar{x}_k, \quad (3)$$

where  $\{i, j, k\} = \{1, 2, 3\}$ .

*Proof.* Consider the following system of equations:

$$\overline{\Gamma(y)} = \theta \bar{y} + p_{21} = \overline{(y, x_1]} + \overline{(y, x_2]} + \overline{(y, x_3]},$$

$$\overline{\Gamma(x_1)} = \theta \bar{x}_1 + p_{21} = \overline{(x_1, y]} + \overline{(x_1, x_2]} + \overline{(x_1, x_3]},$$

$$\overline{\Gamma(x_2)} = \theta \bar{x}_2 + p_{21} = \overline{(x_2, y]} + \overline{(x_2, x_1]} + \overline{(x_2, x_3]},$$

$$\overline{\Gamma(x_3)} = \theta \bar{x}_3 + p_{21} = \overline{(x_3, y]} + \overline{(x_3, x_1]} + \overline{(x_3, x_2]}.$$

In order to derive the required formula (3) it is sufficient to consider the sum of any pair of the last three equations and take into account the other two equations and obvious relations  $\overline{(y, x_i)} = \overline{(x_i, y)} + \bar{x}_i - \bar{y}$ ,  $\overline{(x_i, x_j)} = \overline{(x_j, x_i)} + \bar{x}_i - \bar{x}_j$ .  $\square$

*Proof of Theorem 3.1.* Suppose that  $\theta = \theta_2 = v - 7$ . The right part of (3) must be integer, which implies, for odd  $v$ , that  $\bar{x}_i + \bar{x}_j - \bar{x}_k - \bar{y}$  is even,  $\{i, j, k\} = \{1, 2, 3\}$ .

Without loss of generality, assume that  $y \in P_1$ . Denote by  $t_i$  the number of columns  $\{x_1, x_2, x_3\}$  of  $\Gamma(y)$  such that  $\bar{x}_1 + \bar{x}_2 + \bar{x}_3 = i$  (so that  $t_0 = t_2 = 0$ ). Then

$$t_1 + t_3 = v - 3, t_1 + 3t_3 = p_{11}.$$

It follows that  $t_3 = (p_{11} - v + 3)/2$ ,  $t_1 = (3(v - 3) - p_{11})/2$ . If  $t_1 = 0$  then  $p_{11} = 3(v - 3)$  and  $\Gamma = P_1$ , a contradiction. Suppose that  $M(y)$  does not contain a row of all 1s. Then we may choose a vertex  $x \in \Gamma(y) \cap P_2$  such that  $(x, y]$  contains at least  $t_3 + t_1/3 + 1$  vertices from  $P_1$ . Since, for a column  $\{z_1, z_2, z_3\}$  of  $\Gamma(x)$ , one of the following holds:

$$\{\bar{z}_1, \bar{z}_2, \bar{z}_3\} = \{0, 0, 0\} \text{ or } \{\bar{z}_1, \bar{z}_2, \bar{z}_3\} = \{1, 1, 0\},$$

we see that  $\Gamma(x)$  contains at least  $2(t_3 + t_1/3 + 1)$  vertices from  $P_1$  with equality if and only if  $M(x)$  contains a row of all 0s. This yields

$$2(t_3 + t_1/3 + 1) \leq p_{21},$$

$$(p_{11} - v + 3) + (3(v - 3) - p_{11})/3 + 2 \leq p_{21} = p_{11} - \theta = p_{11} - v + 7,$$

$$(3(v - 3) - p_{11})/3 \leq 2,$$

$$p_{11} \geq 3v - 15.$$

Now if  $y' \in P_2$  and  $M(y')$  does not contain a row of all 0s, then one can show that  $p_{22} \geq 3v - 15$  in the same manner. Taking into account  $p_{11} - p_{21} = \theta_2 = v - 7$ , and  $p_{21} + p_{22} = 3(v - 3)$ , we obtain  $p_{11} + p_{22} = 4v - 16$ . Thus,  $6v - 30 \leq 4v - 16$ , i.e.,  $v \leq 7$ , and Theorem 3.1 now follows from Theorem 2.9.

It remains to exclude the case when  $M(y)$  contains a row of all 1s (or 0s). Suppose that, for every vertex  $y' \in P_2$ , the matrix  $M(y')$  contains a row of all 0s and there is a vertex  $y \in P_1$  such that  $M(y)$  does not contain a row of all 1s. Then it follows from the above that  $p_{11} = 3v - 15$  and thus

$$P = \begin{pmatrix} 3v - 15 & 6 \\ 2v - 8 & v - 1 \end{pmatrix},$$

which contradicts to Lemma 2.4 (for instance,  $\frac{p_{21}}{p_{12}+p_{21}}\binom{v}{3} = \frac{v(v-2)(v-4)}{6}$  is not integer if  $v$  is odd).

Finally, suppose that, for every vertex  $y \in P_1$  (resp.  $y \in P_2$ ), the matrix  $M(y)$  contains a row of all 1s (resp. a row of all 0s). Then, for  $y \in P_2$ , a row of  $M(y)$  that is not of all 0s contains precisely  $p_{21}/2$  1s. On the other hand, this number is  $v - 3 - p_{12}/2 + 1$ . Hence,  $p_{21} + p_{12} = 2v - 4$ . Since  $\sum_{i,j} p_{ij} = 2(3v - 9)$ , we obtain  $p_{11} + p_{22} = 4v - 14$ . We now have  $p_{11} - p_{21} + p_{22} - p_{12} = 4v - 14 - (2v - 4) = 2v - 10 \neq 2\theta_2$ , a contradiction, which completes the proof of Theorem 3.1.  $\square$

**Remark 3.4.** It follows from Lemma 3.3 that

$$0 \leq \overline{(x_i, x_j)} = \overline{(y, x_k]} + \frac{\theta + 1}{2}(\bar{x}_i + \bar{x}_j - \bar{x}_k - \bar{y}) - \bar{x}_k \leq v - 4. \quad (4)$$

Let us consider some examples:

- if  $\bar{y} = 1$  and, for a column  $\{x_1, x_2, x_3\}$  of  $\Gamma(y)$ , we have  $\bar{x}_1 = \bar{x}_2 = \bar{x}_3 = 0$  then  $\overline{(y, x_i]} \geq \frac{\theta+1}{2}$  for  $i = 1, 2, 3$ . Indeed, it directly follows from Lemma 3.3 that  $\overline{(x_j, x_k)} = \overline{(y, x_i]} + \frac{\theta+1}{2}(0 + 0 - 0 - 1) - 0 = \overline{(y, x_i]} - \frac{\theta+1}{2} \geq 0$ .
- if  $v > 8$ ,  $\theta = \theta_2$ , and  $\bar{y} = 1$  then  $M(y)$  does not contain a submatrix of the type

$$\begin{pmatrix} \bar{x}_{1\epsilon} & \bar{x}_{1\delta} \\ \bar{x}_{2\epsilon} & \bar{x}_{2\delta} \\ \bar{x}_{3\epsilon} & \bar{x}_{3\delta} \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Assuming the converse, we see that  $\overline{(x_{1\epsilon}, x_{2\epsilon})} = \overline{(y, x_{3\epsilon}]} + \frac{\theta+1}{2}(1 + 1 - 0 - 1) - 0 = \overline{(y, x_{3\epsilon}]} + \frac{\theta+1}{2}$  so that  $\overline{(y, x_{3\epsilon}]} \leq v - 4 - \frac{\theta+1}{2}$ . On the other hand,  $\overline{(x_{1\delta}, x_{2\delta})} = \overline{(y, x_{3\delta}]} + \frac{\theta+1}{2}(0 + 0 - 1 - 1) - 1 = \overline{(y, x_{3\delta}]} - (\theta + 2)$  and  $\overline{(y, x_{3\epsilon}]} = \overline{(y, x_{3\delta}]} \geq \theta + 2$ , which is impossible as  $v > 8$ .

The arguments similar to the examples above will be used in the proof of Theorem 3.2.  $\square$

We next consider two distinct columns  $\{x_{1\epsilon}, x_{2\epsilon}, x_{3\epsilon}\}$  and  $\{x_{1\delta}, x_{2\delta}, x_{3\delta}\}$ ,  $\epsilon \neq \delta$ , of  $\Gamma(y)$  (assuming that the vertices  $x_{i\epsilon}$  and  $x_{i\delta}$  lay in the same row of  $\Gamma(y)$ ). Then, for  $\{i, j, k\} = \{1, 2, 3\}$ , the vertices  $x_{j\epsilon}$ ,  $x_{k\epsilon}$ ,  $x_{j\delta}$ , and  $x_{k\delta}$  induce a 4-cycle of  $\Gamma(y)$  and, by Lemma 2.1, there is a unique vertex  $z_i \in \Gamma_2(y)$  such that  $\Gamma(y) \cap \Gamma(z_i) = \{x_{j\epsilon}, x_{k\epsilon}, x_{j\delta}, x_{k\delta}\}$ . With this notation, we formulate the following lemma.

**Lemma 3.5.** For  $\{i, j, k\} = \{1, 2, 3\}$  and  $\epsilon \neq \delta$ , the following holds.

$$\overline{(x_{i\delta}, x_{j\delta})} - \overline{(x_{i\epsilon}, x_{k\epsilon})} = \frac{\theta + 1}{2}(\bar{x}_{i\delta} + \bar{z}_k - \bar{z}_j - \bar{x}_{i\epsilon}) + \bar{z}_k - \bar{z}_j - \bar{x}_{j\delta} + \bar{x}_{k\epsilon}. \quad (5)$$

*Proof.* Taking into account the relations:

$$[x_{i\delta}, y] = [x_{i\epsilon}, y], [x_{i\delta}, z_k] = [x_{i\delta}, x_{j\delta}], [x_{i\epsilon}, z_j] = [x_{i\epsilon}, x_{k\epsilon}],$$

the required equality (5) is obtained by applying (2) to the sum of the following equations:

$$\begin{aligned}\overline{\Gamma(x_{i\delta})} &= \overline{(x_{i\delta}, y]} + \overline{(x_{i\delta}, z_j]} + \overline{(x_{i\delta}, z_k]}, \\ -\overline{\Gamma(x_{i\epsilon})} &= -\overline{(x_{i\epsilon}, y]} - \overline{(x_{i\epsilon}, z_j]} - \overline{(x_{i\epsilon}, z_k]}, \\ -\overline{\Gamma(z_j)} &= -\overline{(z_j, x_{i\delta})] - \overline{(z_j, x_{i\epsilon})] - \overline{(z_j, z_k]}}, \\ \overline{\Gamma(z_k)} &= \overline{(z_k, x_{i\delta})] + \overline{(z_k, x_{i\epsilon})] + \overline{(z_k, z_j]}},\end{aligned}$$

which follow from the fact that  $\{z_k, x_{i\epsilon}, x_{i\delta}, z_j\}$  is a maximal 4-clique in  $\Gamma$ .  $\square$

Let us outline the proof of Theorem 3.2. From now on we assume that  $\theta = \theta_2$  and the quotient matrix  $P$  is symmetric. Since  $p_{11} - p_{21} = v - 7$ ,  $p_{12} = p_{21}$  and  $p_{11} + p_{12} = p_{21} + p_{22} = 3(v - 3)$  hold, we have  $p_{11} = p_{22}$  and thus

$$P = \begin{pmatrix} 2v - 8 & v - 1 \\ v - 1 & 2v - 8 \end{pmatrix},$$

and, in addition,  $v \equiv 2 \pmod{4}$ , which is clear from Lemma 2.4.

Further, it follows from [1, Theorem 5] that there is a perfect 2-coloring of  $J(6, 3)$  with quotient matrix

$$\begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix},$$

and the third matrix from (1) provides us with an example of a perfect 2-coloring of  $J(10, 3)$  with symmetric quotient matrix.

Therefore, in the remainder of this section we assume that  $v > 10$  and fix an arbitrary vertex  $y \in P_1$ . The proof consists of two steps. First we shall show that there are exactly four matrices  $M_1, M_2, M_3$ , and  $M_4$  such that  $M(y) \rightsquigarrow \{M_1, M_2, M_3, M_4\}$ , see Lemmas 3.7–3.10. Then, for each  $i = 1, \dots, 4$ , we will show that  $M(y) \sim M_i$  implies that there is a vertex  $x$  of  $\Gamma(y) \cap P_1$  such that  $M(x) \not\rightsquigarrow \{M_1, M_2, M_3, M_4\}$ , see Lemmas 3.11–3.13, and this contradiction will finish the proof.

**Remark 3.6.** Recently Gavriluk, Goryainov, and Mogilnykh showed that there are exactly two nonisomorphic perfect 2-colorings of  $J(10, 3)$  with the same symmetric quotient matrix and eigenvalue  $\theta_2$ . Their arguments were essentially similar to the lemmas below.  $\square$

As above, let  $\{x_{i\delta} | i = 1, 2, 3, \delta = 1, \dots, v - 3\}$  be the vertex set of  $\Gamma(y)$  (so that  $(M(y))_{i\delta} = \bar{x}_{i\delta}$ ). To shorten the notation we also define  $w_i := (y, x_{i\delta}]$ , and  $w_{ij}^\delta := \overline{(x_{i\delta}, x_{j\delta})}$ . Obviously,  $w_i$  and  $w_{ij}^\delta$  are non-negative integers and, moreover,

$$0 \leq w_i \leq v - 3, 0 \leq w_{ij}^\delta = w_{ji}^\delta \leq v - 4.$$

In Lemmas 3.7–3.10 below we assume that  $M(y)$  contains a  $3 \times 2$ -submatrix of a certain type. Let  $\epsilon, \delta$  be the indices of columns of  $M(y)$  that form this submatrix, and, for  $\{i, j, k\} = \{1, 2, 3\}$ , let  $z_i$  be a vertex of  $\Gamma_2(y)$ , such that  $\Gamma(y) \cap \Gamma(z_i) = \{x_{j\epsilon}, x_{k\epsilon}, x_{j\delta}, x_{k\delta}\}$ .

**Lemma 3.7.** *If the matrix  $M(y)$  contains a submatrix of the type*

$$\begin{pmatrix} \bar{x}_{1\epsilon} & \bar{x}_{1\delta} \\ \bar{x}_{2\epsilon} & \bar{x}_{2\delta} \\ \bar{x}_{3\epsilon} & \bar{x}_{3\delta} \end{pmatrix} := \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix},$$

then

$$M(y) \sim M_1 := \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

*Proof.* It follows from Lemma 3.5 that

$$w_{1,2}^\delta - w_{1,3}^\epsilon = \frac{v-4}{2}(\bar{z}_3 - \bar{z}_2) - 1, \text{ and } w_{1,2}^\delta - w_{2,3}^\epsilon = \frac{v-4}{2}(\bar{z}_3 - \bar{z}_1) - 1. \quad (6)$$

From Lemma 3.3 we obtain

$$w_{1,2}^\delta = w_3 + \frac{v-6}{2}, w_{1,3}^\epsilon = w_2 - \frac{v-6}{2} - 1, \text{ and } w_{2,3}^\epsilon = w_1 - \frac{v-6}{2} - 1. \quad (7)$$

Equations (6) and (7) give us the following:

$$w_3 - w_2 = \frac{v-4}{2}(\bar{z}_3 - \bar{z}_2) - v + 4, \quad (8)$$

$$w_3 - w_1 = \frac{v-4}{2}(\bar{z}_3 - \bar{z}_1) - v + 4. \quad (9)$$

In addition, we recall that  $w_1 + w_2 + w_3 = p_{11} = 2v - 8$ . Note that  $\bar{z}_3 - \bar{z}_i \in \{0, \pm 1\}$ ,  $i = 1, 2$ , and let us consider all the possible cases.

If, for instance,  $\bar{z}_3 - \bar{z}_2 = 0$  then equation (8) has only two solutions:

(a)  $w_3 = 1$ ,  $w_2 = v - 3$ , and hence  $w_1 = v - 6$ ,

(b)  $w_3 = 0$ ,  $w_2 = v - 4$ , and hence  $w_1 = v - 4$ .

In Case (a), substituting  $w_3 = 1$  and  $w_1 = v - 6$  into (9) we get  $1 - (v - 6) = (v - 4)(\bar{z}_3 - \bar{z}_1)/2 - v + 4$ . Hence,  $6 = (v - 4)(\bar{z}_3 - \bar{z}_1)$  and  $v = 10$  so that this case may be dropped.

In Case (b), the matrix  $M(y)$  contains a row of all 0s. If  $M(y)$  contains a column, say, with index  $\gamma$ , of all 0s then, by Lemma 3.3, we have  $w_{1,2}^\gamma < 0$  (see also Remark 3.4), a contradiction. Therefore, we conclude that  $M(y)$  is of the required type.

If  $\bar{z}_3 - \bar{z}_2 = -1$  then it follows from (8) that  $w_2 = w_3 + 3(v - 4)/2 \leq v - 3$ . This implies that  $v \leq 6$ .

Finally, if  $\bar{z}_3 - \bar{z}_2 = 1$  then we may also assume that  $\bar{z}_3 - \bar{z}_1 = 1$  and hence  $w_3 - w_2 = w_3 - w_1 = -(v-4)/2$ . Thus  $w_1 = w_2 = 5(v-4)/6$ , and  $w_3 = (v-4)/3$ . It now follows that  $M(y)$  does not contain the following columns:

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

otherwise, for appropriate index  $\gamma$ , by Lemma 3.3, we have  $w_{2,3}^\gamma > v-4$ ,  $w_{2,3}^\gamma < 0$ ,  $w_{1,3}^\gamma > v-4$ ,  $w_{1,3}^\gamma < 0$ , or  $w_{1,2}^\gamma < 0$ , respectively (the last inequality holds for two last columns).

This means that  $M(y)$  is of the type

$$M(y) = \begin{pmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix},$$

which is impossible because  $\Gamma(y)$  then contains at least  $2v-6 > 2v-8$  vertices from  $P_1$ . The lemma is proved.  $\square$

**Lemma 3.8.** *If the matrix  $M(y)$  contains a submatrix of the type*

$$\begin{pmatrix} \bar{x}_{1\epsilon} & \bar{x}_{1\delta} \\ \bar{x}_{2\epsilon} & \bar{x}_{2\delta} \\ \bar{x}_{3\epsilon} & \bar{x}_{3\delta} \end{pmatrix} := \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then

$$M(y) \sim M_2 := \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

*Proof.* It follows from Lemma 3.5 that

$$w_{1,2}^\delta - w_{1,3}^\epsilon = \frac{v-4}{2}(\bar{z}_3 - \bar{z}_2), \quad (10)$$

$$w_{2,1}^\delta - w_{2,3}^\epsilon = \frac{v-4}{2}(\bar{z}_3 - \bar{z}_1) + \frac{v-6}{2}. \quad (11)$$

It follows from Lemma 3.3 that

$$w_{1,2}^\delta = w_3 + \frac{v-6}{2}, w_{1,3}^\epsilon = w_2 + \frac{v-6}{2}, w_{2,3}^\epsilon = w_1 - \frac{v-4}{2}. \quad (12)$$

From (10)–(12) we obtain

$$w_3 - w_2 = \frac{v-4}{2}(\bar{z}_3 - \bar{z}_2) \quad (13)$$

and

$$w_3 - w_1 = \frac{v-4}{2}(\bar{z}_3 - \bar{z}_1) - \frac{v-4}{2}. \quad (14)$$

As before,  $w_1 + w_2 + w_3 = 2v - 8$  and  $\bar{z}_3 - \bar{z}_1 \in \{0, \pm 1\}$ .

Further, it follows from Remark 3.4 (see the second example) that  $M(y)$  does not contain the following columns:

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and, by virtue of the previous lemma, we may also assume that  $M(y)$  does not contain the following columns:

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

apart from those with indices  $\epsilon, \delta$ .

But this yields that  $w_2 = w_3$ .

If  $\bar{z}_3 - \bar{z}_1 = 1$  then  $w_1 = w_2 = w_3 = 2(v-4)/3$ . Since  $w_{1,2}^\delta \leq v-4$ , it follows from (12) that  $w_3 \leq v/2 - 1$ , and hence  $v \leq 10$ .

If  $\bar{z}_3 - \bar{z}_1 = -1$  then (14) gives  $w_1 = v-3$  and  $w_2 = w_3 = 1$ , a contradiction.

Finally, we have  $\bar{z}_3 - \bar{z}_1 = 0$ . Then (14) gives  $w_1 = v-4$ ,  $w_2 = w_3 = (v-4)/2$ . Since  $w_1 > v/2 - 1$ , we conclude that  $M(y)$  does not contain a column  $\gamma$

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

otherwise  $w_{2,3}^\gamma > v-4$ .

Now it is easily seen that  $M(y)$  is of the required type. The lemma is proved.  $\square$

**Lemma 3.9.** *If the matrix  $M(y)$  contains a column of the type*

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

*then  $M(y) \rightsquigarrow \{M_1, M_2\}$ .*

*Proof.* If  $M(y)$  contains a submatrix of one of the following types:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, \text{ or } \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then the lemma follows from the two previous lemmas.

Therefore, we suppose that  $M(y)$  contains exactly one column of the type

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

and it now follows from Remark 3.4 that  $M(y)$  does not contain a column

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Hence  $M(y)$  may only contain the following columns (where exponents mean multiplicities of columns):

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}^1, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^a, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^b, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}^c, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}^d,$$

where we have  $1 + a + b + c + d = v - 3$ ,  $2 + a + b + 3c = 2v - 8$ , and these equations give  $c \geq v/2 - 3$ .

Let us consider a submatrix of  $M(y)$  of the type

$$\begin{pmatrix} \bar{x}_{1\epsilon} & \bar{x}_{1\delta} \\ \bar{x}_{2\epsilon} & \bar{x}_{2\delta} \\ \bar{x}_{3\epsilon} & \bar{x}_{3\delta} \end{pmatrix} := \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that  $w_1 = 1 + a + c$ ,  $w_2 = 1 + b + c$ ,  $w_3 = c$ .

It follows from Lemma 3.5 that

$$w_{1,2}^\delta - w_{1,3}^\epsilon = \frac{v-4}{2}(\bar{z}_3 - \bar{z}_2), \text{ and } w_{1,2}^\delta - w_{2,3}^\epsilon = \frac{v-4}{2}(\bar{z}_3 - \bar{z}_1). \quad (15)$$

From Lemma 3.3 we obtain

$$w_{1,2}^\delta = w_3 + \frac{v-6}{2}, w_{1,3}^\epsilon = w_2 - 1, \text{ and } w_{2,3}^\epsilon = w_1 - 1, \quad (16)$$

and thus (15) and (16) give

$$w_3 - w_2 = \frac{v-4}{2}(\bar{z}_3 - \bar{z}_2) - \frac{v-4}{2}, \quad (17)$$

$$w_3 - w_1 = \frac{v-4}{2}(\bar{z}_3 - \bar{z}_1) - \frac{v-4}{2}. \quad (18)$$

Let us consider all the possible values of  $\bar{z}_3 - \bar{z}_2$  (our objective right now is to obtain a contradiction for every value).



If  $\bar{z}_3 - \bar{z}_2 = -1$  then it follows from (17) that  $w_2 = v - 3$ , and  $w_3 = 1$ , which contradicts  $c \geq v/2 - 3$  and  $v > 10$ .

If  $\bar{z}_3 - \bar{z}_2 = 1$  then it follows from (17) that  $w_2 = w_3$ , which is impossible.

Finally, we may suppose that  $\bar{z}_3 - \bar{z}_2 = \bar{z}_3 - \bar{z}_1 = 0$ . It follows from (17) and (18) that  $w_1 = w_2 = 5(v - 4)/6$ , and  $w_3 = (v - 4)/3$ . Since  $w_3 \geq v/2 - 3$ , we have  $v \leq 10$ , a contradiction. The lemma is proved.  $\square$

**Lemma 3.10.** *If  $M(y) \not\sim \{M_1, M_2\}$  then  $M(y) \rightsquigarrow \{M_3, M_4\}$ , where*

$$M_3 := \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, M_4 := \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

*Proof.* Since  $M(y) \not\sim M_1$  and  $M(y) \not\sim M_2$ , we see that  $M(y)$  may only contain the following columns (where exponents mean multiplicities of columns):

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}^a, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^{b_1}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^{b_2}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^{b_3}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}^c.$$

We have  $a + b_1 + b_2 + b_3 + c = v - 3$ ,  $3a + b_1 + b_2 + b_3 = 2v - 8$ . If  $b_i > 0$  then it follows from Lemma 3.3 that  $a + b_i \geq v - 5$ . Moreover, if  $b_i b_j > 0$ ,  $i \neq j$ , then it is easily seen that  $a$  is at least  $2(v - 5) - (v - 3) = v - 7$ . Further, then  $3a + b_1 + b_2 + b_3 \geq 2(v - 5) + (v - 7)$  and this yields  $v \leq 9$ . Therefore, without loss of generality, we may assume that  $b_2 = b_3 = 0$ . If, in addition,  $b_1 = 0$ , then  $M(y) \sim M_4$ .

If  $b_1 > 0$  then we have already noted that  $a + b_1 \geq v - 5$  so that  $c \in \{0, 1, 2\}$ . By the above, we have  $a + b_1 + c = v - 3$ ,  $3a + b_1 = 2v - 8$  and hence  $2a = v - 5 + c$ . Since  $v$  is even, we see that  $c$  is odd, i.e.,  $c = 1$  and  $M(y) \sim M_3$ . The lemma is proved.  $\square$

Our next step is to show that, for every type of  $M(y)$ , there is a vertex  $x \in \Gamma(y) \cap P_1$  such that  $M(x) \not\sim \{M_1, M_2, M_3, M_4\}$ , which contradicts the lemmas above. We call such a vertex  $x$  *bad*.

**Lemma 3.11.** *Let  $M(y) \sim M_3$ . Then  $\Gamma(y)$  contains a bad vertex.*

*Proof.* Under the assumptions of the lemma, suppose that

$$\begin{pmatrix} \bar{x}_{1,1} & \cdots & \bar{x}_{1,a} & \bar{x}_{1,a+1} & \cdots & \bar{x}_{1,v-4} & \bar{x}_{1,v-3} \\ \bar{x}_{2,1} & \cdots & \bar{x}_{2,a} & \bar{x}_{2,a+1} & \cdots & \bar{x}_{2,v-4} & \bar{x}_{2,v-3} \\ \bar{x}_{3,1} & \cdots & \bar{x}_{3,a} & \bar{x}_{3,a+1} & \cdots & \bar{x}_{3,v-4} & \bar{x}_{3,v-3} \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\ 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where  $\bar{x}_{2,a} = 1$ ,  $\bar{x}_{2,a+1} = 0$ ,  $a = (v - 4)/2$ .

Let us consider a vertex  $x_{2,1}$ . The matrix  $M(x_{2,1})$  has the following form:

$$\begin{pmatrix} \bar{x}_{1,1} & \bar{z}_{3,2} & \bar{z}_{3,3} & \cdots & \bar{z}_{3,v-3} \\ \bar{y} & \bar{x}_{2,2} & \bar{x}_{2,3} & \cdots & \bar{x}_{2,v-3} \\ \bar{x}_{3,1} & \bar{z}_{1,2} & \bar{z}_{1,3} & \cdots & \bar{z}_{1,v-3} \end{pmatrix},$$

where, for  $\{i, j\} = \{1, 3\}$  and  $\delta = 2, \dots, v-3$ , a vertex  $z_{i\delta} \in \Gamma_2(y)$  and  $\Gamma(y) \cap \Gamma(z_{i\delta}) = \{x_{j1}, x_{j\delta}, x_{2\delta}, x_{2,1}\}$ .

Applying Lemmas 3.3 and 3.5 to a pair of the first and the  $\delta$ th columns of  $M(y)$  gives an equation from (5) with respect to  $\bar{z}_{1\delta} - \bar{z}_{3\delta}$ . Obviously,  $\bar{z}_{1\delta} - \bar{z}_{3\delta} \in \{-1, 0, 1\}$ , and if  $\bar{z}_{1\delta} - \bar{z}_{3\delta} \neq 0$ , we can derive the values  $\bar{z}_{1\delta}$  and  $\bar{z}_{3\delta}$ .

Let us consider an example with the first two columns of  $M(y)$ :

$$\begin{pmatrix} \bar{x}_{1,1} & \bar{x}_{1,2} \\ \bar{x}_{2,1} & \bar{x}_{2,2} \\ \bar{x}_{3,1} & \bar{x}_{3,2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

We have  $\overline{(y, x_{1,1})} = v-4$  and  $\overline{(y, x_{3,2})} = (v-4)/2$  so that  $\overline{(x_{2,1}, x_{3,1})} = v-5$ , and  $\overline{(x_{2,2}, x_{1,2})} = (v-6)/2$  by Lemma 3.3. Now Lemma 3.5 gives:

$$\overline{(x_{2,2}, x_{1,2})} - \overline{(x_{2,1}, x_{3,1})} = \frac{v-6}{2}(\bar{x}_{2,2} + \bar{z}_{3,2} - \bar{z}_{1,2} - \bar{x}_{2,1}) + \bar{z}_{3,2} - \bar{z}_{1,2} - \bar{x}_{1,2} + \bar{x}_{3,1},$$

$$\text{i.e., } (v-6)/2 - v + 5 = \frac{v-6}{2}(\bar{z}_{3,2} - \bar{z}_{1,2}) + \bar{z}_{3,2} - \bar{z}_{1,2},$$

which leads to  $z_{3,2} = 0$  and  $z_{1,2} = 1$ .

Solving similar equations for  $\delta = 2, \dots, v-3$ , we have

$$\begin{pmatrix} \bar{x}_{1,1} & \bar{z}_{3,2} & \cdots & \bar{z}_{3,a} & \bar{z}_{3,a+1} & \cdots & \bar{z}_{3,v-4} & \bar{z}_{3,v-3} \\ \bar{y} & \bar{x}_{2,2} & \cdots & \bar{x}_{2,a} & \bar{x}_{2,a+1} & \cdots & \bar{x}_{2,v-4} & \bar{x}_{2,v-3} \\ \bar{x}_{3,1} & \bar{z}_{1,2} & \cdots & \bar{z}_{1,a} & \bar{z}_{1,a+1} & \cdots & \bar{z}_{1,v-4} & \bar{z}_{1,v-3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 & \spadesuit & \cdots & \spadesuit & 0 \\ 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & \spadesuit & \cdots & \spadesuit & 1 \end{pmatrix},$$

where  $\spadesuit$  means that the corresponding value of  $\bar{z}_{1,\delta} - \bar{z}_{3,\delta}$  is 0 (so that the two symbols  $\spadesuit$  in one column mean the same value).

We see that  $M(x_{2,1})$  simultaneously contains a column of all 1s and a submatrix

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix},$$

contrary to Lemma 3.7. The lemma is proved.  $\square$

**Lemma 3.12.** *Let  $M(y) \sim M_2$ . Then  $\Gamma(y)$  contains a bad vertex.*

*Proof.* Under the assumptions of the lemma, suppose that

$$\begin{pmatrix} \bar{x}_{1,1} & \cdots & \bar{x}_{1,a} & \bar{x}_{1,a+1} & \bar{x}_{1,a+2} & \bar{x}_{1,a+3} & \cdots & \bar{x}_{1,v-4} & \bar{x}_{1,v-3} \\ \bar{x}_{2,1} & \cdots & \bar{x}_{2,a} & \bar{x}_{2,a+1} & \bar{x}_{2,a+2} & \bar{x}_{2,a+3} & \cdots & \bar{x}_{2,v-4} & \bar{x}_{2,v-3} \\ \bar{x}_{3,1} & \cdots & \bar{x}_{3,a} & \bar{x}_{3,a+1} & \bar{x}_{3,a+2} & \bar{x}_{3,a+3} & \cdots & \bar{x}_{3,v-4} & \bar{x}_{3,v-3} \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & \cdots & 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where  $\bar{x}_{2,a} = 1, \bar{x}_{2,a+1} = 0, a = (v-4)/2 - 1$ .

Again let us consider the matrix  $M(x_{2,1})$ :

$$\begin{pmatrix} \bar{x}_{1,1} & \bar{z}_{3,2} & \bar{z}_{3,3} & \cdots & \bar{z}_{3,v-3} \\ \bar{y} & \bar{x}_{2,2} & \bar{x}_{2,3} & \cdots & \bar{x}_{2,v-3} \\ \bar{x}_{3,1} & \bar{z}_{1,2} & \bar{z}_{1,3} & \cdots & \bar{z}_{1,v-3} \end{pmatrix},$$

where, for  $\{i, j\} = \{1, 3\}$  and  $\delta = 2, \dots, v-3$ , a vertex  $z_{i\delta} \in \Gamma_2(y)$  and  $\Gamma(y) \cap \Gamma(z_{i\delta}) = \{x_{j1}, x_{j\delta}, x_{2\delta}, x_{2,1}\}$ .

Analysis similar to that in the proof of Lemma 3.11 shows that

$$\begin{pmatrix} \bar{x}_{1,1} & \bar{z}_{3,2} & \cdots & \bar{z}_{3,a} & \bar{z}_{3,a+1} & \bar{z}_{3,a+2} & \bar{z}_{3,a+3} & \cdots & \bar{z}_{3,v-4} & \bar{z}_{3,v-3} \\ \bar{y} & \bar{x}_{2,2} & \cdots & \bar{x}_{2,a} & \bar{x}_{2,a+1} & \bar{x}_{2,a+2} & \bar{x}_{2,a+3} & \cdots & \bar{x}_{2,v-4} & \bar{x}_{2,v-3} \\ \bar{x}_{3,1} & \bar{z}_{1,2} & \cdots & \bar{z}_{1,a} & \bar{z}_{1,a+1} & \bar{z}_{1,a+2} & \bar{z}_{1,a+3} & \cdots & \bar{z}_{1,v-4} & \bar{z}_{1,v-3} \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 1 & \spadesuit & \cdots & \spadesuit & 0 \\ 1 & 1 & \cdots & 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \spadesuit & \cdots & \spadesuit & 1 \end{pmatrix},$$

where the two symbols  $\spadesuit$  in a column mean the same value (0 or 1), and the rest of the proof runs as before.  $\square$

**Lemma 3.13.** *Let  $M(y) \sim M_1$ . Then  $\Gamma(y)$  contains a bad vertex.*

*Proof.* Under the assumptions of the lemma, suppose that

$$\begin{pmatrix} \bar{x}_{1,1} & \cdots & \bar{x}_{1,v-5} & \bar{x}_{1,v-4} & \bar{x}_{1,v-3} \\ \bar{x}_{2,1} & \cdots & \bar{x}_{2,v-5} & \bar{x}_{2,v-4} & \bar{x}_{2,v-3} \\ \bar{x}_{3,1} & \cdots & \bar{x}_{3,v-5} & \bar{x}_{3,v-4} & \bar{x}_{3,v-3} \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 & 1 & 0 \\ 1 & \cdots & 1 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

Analysis similar to that in the proof of Lemma 3.11 shows that the matrix  $M(x_{1,1})$  has the following form:

$$\begin{pmatrix} \bar{y} & \bar{x}_{1,2} & \cdots & \bar{x}_{1,v-5} & \bar{x}_{1,v-4} & \bar{x}_{1,v-3} \\ \bar{x}_{2,1} & \bar{z}_{3,2} & \cdots & \bar{z}_{3,v-5} & \bar{z}_{3,v-4} & \bar{z}_{3,v-3} \\ \bar{x}_{3,1} & \bar{z}_{2,2} & \cdots & \bar{z}_{2,v-5} & \bar{z}_{2,v-4} & \bar{z}_{2,v-3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & \spadesuit & \cdots & \spadesuit & 0 & 0 \\ 0 & \spadesuit & \cdots & \spadesuit & 1 & 0 \end{pmatrix},$$

where, for  $\{i, j\} = \{2, 3\}$  and  $\delta = 2, \dots, v-3$ , a vertex  $z_{i\delta} \in \Gamma_2(y)$  and  $\Gamma(y) \cap \Gamma(z_{i\delta}) = \{x_{j1}, x_{j\delta}, x_{1\delta}, x_{1,1}\}$ , and the two symbols  $\spadesuit$  in a column mean the same value (0 or 1).

We see that  $M(x_{1,1})$  contains a submatrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and Lemma 3.8 implies that  $M(x_{1,1}) \sim M_2$ , contrary to Lemma 3.12. The lemma is proved.  $\square$

Let us complete the proof of Theorem 3.2. Lemmas 3.11–3.13 now imply that  $M(y) \sim M_4$ . A similar conclusion (with 1 replaced by 0) can be drawn for a vertex of  $P_2$  by virtue of symmetry of quotient matrix  $P$ . We leave it to the reader to verify that this leads to a contradiction, which proves Theorem 3.2.  $\square$

#### 4. CONCLUSION

In this paper, we have classified perfect 2-colorings of  $J(v, 3)$  with odd  $v$ , and gave a partial result on perfect 2-colorings of  $J(v, 3)$ , when  $v$  is even, with specific quotient matrix.

In the forthcoming paper, we will extend our approach in order to classify all the realizable quotient matrices of perfect 2-colorings of  $J(v, 3)$  with even  $v$ . In fact, if  $P$  is a quotient matrix of one of them with eigenvalue  $\theta_2$  then  $P$  is one of the matrices (1). Moreover, if  $v > 10$  then the corresponding perfect 2-coloring is unique (up to automorphisms of the graph).

Unfortunately, it seems that our approach cannot be directly generalized to study perfect 2-colorings of  $J(v, k)$ ,  $k > 3$ . (Roughly speaking, a generalization of Lemma 3.3 would deal with a system of  $k$  linear equations with  $\binom{k}{2}$  unknowns.)

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